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Stationary Turbulent Closure via the Hopf Functional Equation

Hubert H. Shen[1] **and Alan A. Wray**[2]

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Exact closed-form solutions are exhibited for the Hopf equation for stationary incompressible 3D Navier–Stokes flow, for the cases of homogeneous forced flow (including a solution with depleted nonlinearity) and inhomogeneous flow with arbitrary boundary conditions. This provides an exact method for computing two- and higher-point moments, given the mean flow.

KEY WORDS: Navier–Stokes turbulence; closure; generating functional; Hopf equation.

1. INTRODUCTION

"It is commonly accepted that turbulent flow is necessarily statistical in nature. Hopf formulated an equation governing the probability function for such flows,[1] but so far no genuinely physical explicit solutions have been obtained...."[2] Thus, despite the fact that the Hopf approach has been characterized by some as "the most compact formulation of the general turbulence problem"[3] and even "the only exact formulation in the entire field of turbulence,"[4] its actual usefulness in predicting statistics has until now been extremely limited by the lack of explicit solutions. By applying the Navier–Stokes equation to the moment-generating functional for the velocity, the Hopf approach transforms a nonlinear differential equation describing a single flow realization to a linear functional-integrodifferential equation governing an ensemble of flows. However, in the absence of a general method for solving such equations, results have until now been mostly of a formal nature.[5,6]

[1] Center for Turbulence Research, NASA-Ames/Stanford University.
[2] NASA-Ames Research Center.

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It is our purpose here to exhibit explicit solutions of the stationary Hopf equation and begin to explore their computational possibilities. The motivation is to circumvent the infinite hierarchy of coupled equations for the velocity moments and obtain an exact closure of the steady-state 3D Navier–Stokes equations, without modeling assumptions or truncation. In Section 2, we review the Hopf formulation of the Navier–Stokes equation. In Section 3.1, we display and discuss a stationary homogeneous solution for 2D flow. In Section 3.2, we show how depletion of nonlinearity may arise for 3D forced homogeneous flow. Section 3.3 considers the general 3D forced case, while Section 3.4 derives a method for closing the 3D unforced equations with arbitrary boundary conditions. We conclude with future plans.

2. REVIEW: HOPF EQUATION

Recall[1] the definition of the Hopf functional

$$\Phi[\mathbf{f}(\mathbf{x})] \equiv \left\langle \exp\left(i \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \right) \right\rangle \tag{2.1}$$

Its input is an arbitrary nonrandom time-independent "conjugate," "dummy," or "test" function $\mathbf{f}(\mathbf{x})$; the values of $\mathbf{f}$ at *all* $\mathbf{x}$ are required. The output is a number independent of $\mathbf{x}$, namely, the ensemble average (over the velocity field $\mathbf{u}(\mathbf{x})$ at *all* points, with probability density functional $P[\mathbf{u}(\mathbf{x})]$) of the quantity within the brackets.

If one defines the functional derivative

$$\frac{\delta\Phi[\mathbf{f}(\mathbf{x})]}{\delta f_j(\mathbf{x}')} \equiv \lim_{\varepsilon \to 0} \left\{ \frac{\Phi[\mathbf{f}(\mathbf{x}) + \hat{\jmath}\varepsilon\delta(\mathbf{x} - \mathbf{x}')] - \Phi[\mathbf{f}(\mathbf{x})]}{\varepsilon} \right\} \tag{2.2}$$

(which depends upon $\mathbf{x}'$, but not $\mathbf{x}$; $\hat{\jmath}$ is a unit vector), then one may readily verify that

$$\left[\frac{\delta\Phi[\mathbf{f}(\mathbf{x})]}{\delta f_j(\mathbf{x}')} \right]_{\mathbf{f}=0} = \langle (i)\, u_j(\mathbf{x}') \rangle, \qquad \left[\frac{\delta^2\Phi[\mathbf{f}(\mathbf{x})]}{\delta f_j(\mathbf{x})\, \delta f_k(\mathbf{x}')} \right]_{\mathbf{f}=0} = \langle (i)^2\, u_j(\mathbf{x})\, u_k(\mathbf{x}') \rangle \tag{2.3}$$

etc. This arises from identities such as

$$\lim_{\varepsilon \to 0} \left\{ \frac{\exp[i\varepsilon \int d\mathbf{x}\, u_j(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}')] - 1}{\varepsilon} \right\} = (i)\, u_j(\mathbf{x}') \tag{2.3$'$}$$

In other words, Φ is the characteristic functional or moment-generating functional for the velocity field $\mathbf{u}(\mathbf{x})$, containing all equal-time statistical information about $\mathbf{u}(\mathbf{x})$. Intermittency is included in this description insofar as it can be captured in the higher moments of velocity.

If one defines the inverse functional Fourier transform

$$\tilde{\Phi}[\mathbf{v}(\mathbf{x})] \equiv \int \left[\exp\left(-i \int d\mathbf{x}\, \mathbf{f} \cdot \mathbf{v}\right)\right] \Phi[\mathbf{f}(\mathbf{x})] \prod_{\mathbf{x}} d\mathbf{f}(\mathbf{x}) \tag{2.4}$$

where the outer integral is over all values of $\mathbf{f}$ evaluated at all points in space $\mathbf{x}$, then one may verify that $\tilde{\Phi}[\mathbf{v}(\mathbf{x})]$ is just the probability density functional $P[\mathbf{v}(\mathbf{x})]$ for the velocity field $\mathbf{v}(\mathbf{x})$. This result is expected because, for discrete $\mathbf{x}$, the functional derivative and functional Fourier transform reduce to the conventional partial derivative and multivariable Fourier transform, respectively. Furthermore, as desired, the result does not depend on $\mathbf{v}$ being independent at different points $\mathbf{x}$, i.e., it does not require P to factor into a product of probability distributions for $\mathbf{v}$ at each $\mathbf{x}$.

The time evolution of Φ is given by

$$\partial_t \left\langle \exp\left(i \int d\mathbf{x}\, \mathbf{f} \cdot \mathbf{u}\right)\right\rangle = \left\langle i \int d\mathbf{x}\, \mathbf{f} \cdot \partial_t \mathbf{u} \exp\left(i \int d\mathbf{x}\, \mathbf{f} \cdot \mathbf{u}\right)\right\rangle \tag{2.5}$$

where $\partial_t \mathbf{u}$ is given by the Navier–Stokes equation. Now $\mathbf{f}$ may be decomposed[3] into two components, namely, a gradient term ∇g and a remainder $\bar{\mathbf{f}}$. These two components will be orthogonal functions in the sense that $\int d\mathbf{x}\, \bar{\mathbf{f}} \cdot \nabla g = 0$ if $\bar{\mathbf{f}}$ is chosen to be solenoidal and have vanishing normal component at the boundary (as one may verify by integrating by parts). But these are just the conditions satisfied by $\mathbf{u}$. Hence $\int d\mathbf{x}\, \mathbf{u} \cdot \nabla g = 0$ and $\mathbf{f}$ may be replaced by $\bar{\mathbf{f}}$ in all of the above equations. The advantage of this replacement is that it eliminates the pressure contribution to Eq. (2.5). Also, because $\bar{\mathbf{f}}$ is solenoidal, the number of independent scalar fields which comprise it has been reduced from 3 to 2.

The equation of motion then becomes

$$\partial_t \left\langle \exp\left(i \int d\mathbf{x}\, \bar{\mathbf{f}} \cdot \mathbf{u}\right)\right\rangle = \left\langle i \int d\mathbf{x}\, \bar{\mathbf{f}} \cdot \partial_t \mathbf{u} \exp\left(i \int d\mathbf{x}\, \bar{\mathbf{f}} \cdot \mathbf{u}\right)\right\rangle \tag{2.6}$$

$$= \left\langle i \int d\mathbf{x}\, \bar{\mathbf{f}} \cdot (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) \exp\left(i \int d\mathbf{x}\, \bar{\mathbf{f}} \cdot \mathbf{u}\right)\right\rangle \tag{2.6'}$$

which, using Eq. (2.3), becomes the Hopf equation:

$$\frac{\partial \Phi}{\partial t} = \int d\mathbf{x}\, \bar{f}_j \left(i \frac{\partial}{\partial x_k} \frac{\delta^2}{\delta \bar{f}_k\, \delta \bar{f}_j} + \nu \nabla^2 \frac{\delta}{\delta \bar{f}_j} \right) \Phi \tag{2.7}$$

where repeated indices are summed over and the $\bar{f}$'s are understood as having the argument $\mathbf{x}$ unless otherwise noted.

3. RESULTS

3.1. Steady-State Solutions

To find steady-state solutions, let us rewrite this equation as

$$\frac{\partial \Phi}{\partial t} = \int d\mathbf{x}\, \bar{f}_j \frac{\partial}{\partial x_k} \left[\left(i \frac{\delta}{\delta \bar{f}_k} + \nu \frac{\partial}{\partial x_k} \right) \frac{\delta \Phi}{\delta \bar{f}_j} \right] \tag{3.1}$$

Note that the expression inside the parentheses is essentially the kernel for a "wave" equation in which $\bar{f}_k$ and x_k play the role of position and time, respectively. Hence we will have stationarity if, for example,

$$\frac{\delta \Phi}{\delta \bar{f}_j} = G_j \left(\frac{i}{\nu} \mathbf{x} - \int d\mathbf{x}'\, \bar{\mathbf{f}}(\mathbf{x}') \right) \tag{3.2}$$

where G_j is an arbitrary function and the integral is over all space.

The first term inside the parentheses is acted upon by the viscous term of the Navier–Stokes equation, while the second term inside the parentheses is acted upon by the convective term. The steady-state balance between the two terms corresponds (in the parlance of a harmonic-oscillator formulation[7] of the Navier–Stokes equations) to a state in which creation and annihilation processes balance, i.e., an oscillator at its apogee or perigee. (This condition of balance distinguishes our solution from the Lewis and Kraichnan[8] solution of the Hopf equation for the time-dependent but *linearized* Navier–Stokes equation.) This particular solution can only take on physical significance after we specify some explicit external forcing and/or boundary conditions which can input energy. Until then, it [and its generalization (3.5)] may be viewed as useful paradigms for more realistic solutions, to be discussed in later sections.

More generally,

$$\frac{\delta \Phi}{\delta \bar{f}_j} = G_j \left(\frac{i}{\nu} H(\mathbf{x}) - \int d\mathbf{x}'\, \bar{\mathbf{f}}(\mathbf{x}') \cdot \nabla H(\mathbf{x}') \right) \tag{3.3}$$

satisfies the steady-state Hopf equation. However, by the construction of $\bar{\mathbf{f}}$, the second term inside the parentheses vanishes unless one restricts oneself to flows in which the pressure gradient may be neglected in the equations of motion (which would be the "opposite" of inviscid Beltrami flows in the sense that the gradient of kinetic energy would not be balanced by the pressure gradient, but by the viscous and Coriolis forces). An example would be 2D flow,[9] in which the equation of motion (as derived from the vorticity equation) for the joint velocity–vorticity characteristic functional

$$\Phi[\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})] \equiv \left\langle \exp\left(i \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x})\right)\right\rangle \tag{3.4}$$

has neither pressure gradient nor vortex-stretching terms. Its steady-state first functional derivative would then be given by

$$\frac{\delta \Phi}{\delta g_j} = G_j \left(\frac{i}{v} H(x) - \int d\mathbf{x}'\, \mathbf{f}(\mathbf{x}') \cdot \nabla H(\mathbf{x}')\right) \tag{3.5}$$

An analogous result would hold for a passive diffusing scalar advected by a steady flow field. Equation (3.3) would also constitute a steady-state solution of the Hopf equation for the *one*-dimensional fluid (Burger's equation).

The possibility of attractors (e.g., soliton or shocklike solutions) for our functional "wave" equation would be intriguing, since attractors at $\mathbf{f} = 0$ would correspond to stable statistical solutions. However, because the Hopf equation for Navier–Stokes is linear in Φ and nondispersive (constant functional "group" velocity), there does not appear to be any mechanism for steepening or evolution of wavefronts and stability must be determined by other means.

One could in principle perform a functional integration upon (3.3) or (3.5) to obtain Φ. However, for computational purposes it is easier to work directly with the first functional derivative of Φ, as we will see. Note that the joint functional (3.4) is overcomplete in the sense that the $\mathbf{f}$ and $\mathbf{g}$ in (3.4) are not independent, because $\mathbf{u}$ and $\boldsymbol{\omega}$ are not independent. If they were independent, one could immediately integrate (3.5) to obtain

$$\Phi = \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{g}(\mathbf{x}) \cdot \mathbf{G}\left(\frac{i}{v} H(\mathbf{x}) - \int d\mathbf{x}'\, \mathbf{f}(\mathbf{x}') \cdot \nabla H(\mathbf{x}')\right) \tag{3.6}$$

whose velocity moments would all vanish, contrary to reality. Alternatively, working in $\mathbf{k}$ space, one may verify that functional derivatives with respect to $\tilde{\mathbf{f}}(\mathbf{k})$ are equivalent to those with respect to $i\mathbf{k} \times \tilde{\mathbf{g}}(\mathbf{k})$.

For the particular choice

$$H(\mathbf{x}) = \exp(i\mathbf{k} \cdot \mathbf{x}) \qquad G_j(x) = a_j \ln x \tag{3.7}$$

the statistics generated by Φ turn out to be homogeneous in space. To see this, consider

$$\frac{\delta \Phi}{\delta g_j} = \sum_m \left(\frac{a_{mj}}{v k_{mj}}\right) \ln \left[\frac{\exp(i\mathbf{k}_m \cdot \mathbf{x})}{v} - \int d\mathbf{x}' \, \mathbf{f}(\mathbf{x}') \cdot \mathbf{k}_m \exp(i\mathbf{k}_m \cdot \mathbf{x}')\right] \tag{3.8}$$

Then, taking one more functional derivative, we obtain

$$\frac{\delta^2 \Phi[\mathbf{f}(\mathbf{x})]}{\delta g_j(\mathbf{x}) \, \delta f_k(\mathbf{x}')} = \sum_m \left(\frac{a_{mj}}{v k_{mj}}\right) \frac{-k_{mk} \exp(i\mathbf{k}_m \cdot \mathbf{x}')}{z_m(\mathbf{x})} \tag{3.9}$$

where

$$z_m(\mathbf{x}) \equiv \left(\frac{\exp(i\mathbf{k}_m \cdot \mathbf{x})}{v} - \int d\mathbf{x}' \, \mathbf{f}(\mathbf{x}') \cdot \mathbf{k}_m \exp[i\mathbf{k}_m \cdot \mathbf{x}']\right) \tag{3.10}$$

Hence, setting $\mathbf{f} = 0$ and using Eq. (2.3) yields

$$\langle \omega_j(\mathbf{x}) \, u_j(\mathbf{x}') \rangle = \sum_m a_{mj} \exp[i\mathbf{k}_m \cdot (\mathbf{x} - \mathbf{x}')] \tag{3.11}$$

which exhibits homogeneity.

In order to construct velocity moments which are real, note that the complex conjugate of (3.8) is *not* a solution of (2.7), but rather of the complex conjugate of (2.7). However,

$$\frac{\delta \Phi}{\delta g_j} = \sum_m \left(\frac{a_{mj}}{v k_{mj}}\right) \ln \left[\frac{-\exp(-i\mathbf{k}_m \cdot \mathbf{x})}{v} - \int d\mathbf{x}' \, \mathbf{f}(\mathbf{x}') \cdot \mathbf{k}_m \exp(-i\mathbf{k}_m \cdot \mathbf{x}')\right] \tag{3.12}$$

is a solution of (2.7). Linearity allows us to choose any linear combination of (3.8) and (3.12) as our solution; we choose the difference between the two expressions, since this has the additional property that its integral with respect to the Fourier component of **f** converges. This difference solution has the structure function

$$\langle \omega_j(\mathbf{x}) \, u_j(\mathbf{x}') \rangle = 2 \sum_m a_{mj} \cos[\mathbf{k}_m \cdot (\mathbf{x} - \mathbf{x}')] \tag{3.13}$$

which is real, as desired.

We may in general add another term

$$J \equiv i \int d\mathbf{x}' \, (C\mathbf{x}' + \mathbf{D}) \cdot \mathbf{g}(\mathbf{x}') \tag{3.14}$$

to Φ, since the Hopf equation is linear in Φ and quadratic in spatial and functional derivatives. C is a constant matrix and $\mathbf{D}$ is a constant vector, to be determined. Then the mean vorticity becomes

$$\langle \omega_j(\mathbf{x}) \rangle = \pi(\ln \nu) \sum_m \left(\frac{a_{mj}}{\nu k_{mj}} \right) + 2 \sum_m \left(\frac{a_{mj}}{\nu k_{mj}} \right) \mathbf{k}_m \cdot \mathbf{x} + C\mathbf{x} + \mathbf{D} \tag{3.15}$$

Incompressibility and the prevailing mean vorticity and vorticity gradient determine $\mathbf{D}$ and C, respectively. One may also match the homogeneous intensity $\langle u^2 \rangle$ of the velocity fluctuations by adding a term

$$J' \equiv E \left| \int d\mathbf{x}' \, \mathbf{f}(\mathbf{x}') \right|^2 \tag{3.16}$$

to Φ, where E is a constant. An analogous term may be added to match the mean square vorticity and in order to satisfy realizability (e.g., non-negative variance of vorticity). [Although the individual contributions (3.8), (3.12), (3.14), (3.16), etc., which make up the solution may not be realizable, their sum *does* constitute a realizable solution.]

The source of energy in this case is the implicit force which maintains the mean vorticity profile. Hence this particular solution may be viewed as resembling decaying turbulence at high Reynolds number and small length scales (with effective forcing by the mean flow). In order to obtain a more satisfactory solution which achieves mathematical *and* physical stationarity (incorporating a more explicit source of energy), we modify our approach, as follows.

3.2. Depletion of Nonlinearity

Let us consider the 3D case (i.e., restore vortex-stretching to the equations) and add explicit external forcing $\mathbf{F}(\mathbf{x})$. If we consider the joint velocity–vorticity–force characteristic functional

$$\Phi[\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x})] \equiv \left\langle \exp\left[i \int_{-\infty}^{\infty} d\mathbf{x} \, \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) \right] \right\rangle \tag{3.17}$$

then the conditions that viscosity and forcing balance (implying that stretching and advection balance, in order to achieve stationarity) take the respective forms

$$\nabla \times \nu \nabla^2 \frac{\delta \Phi}{\delta \mathbf{f}(\mathbf{x})} = -\nabla \times \frac{\delta \Phi}{\delta \mathbf{h}(\mathbf{x})} \tag{3.18}$$

$$\nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \frac{\delta \Phi}{\delta \mathbf{g}(\mathbf{x})} \right) = 0 \tag{3.19}$$

This is a special case of general stationarity, with the "Eulerization" constraint that $\nabla \times (\mathbf{u} \times \boldsymbol{\omega})$ vanishes everywhere. Technically, this constraint is only *weakly* or statistically imposed, i.e., only its ensemble average with any moment of velocity is required to vanish.[10] This constraint is motivated by a suggestion by Moffatt[11] and by recent experimental, numerical, and analytical work[12] indicating that decaying turbulent flows tend to spend a significant portion of their time in the vicinity of fixed points of the Euler equation, in which $\mathbf{u} \times \boldsymbol{\omega} = \nabla(P + \frac{1}{2}u^2)$. This amounts to a depletion of nonlinearity, since the total nonlinear term is the solenoidal part of $\mathbf{u} \times \boldsymbol{\omega}$. This is directly relevant to issues of turbulent drag reduction and coherent structures, since both can arise from reduced enstrophy production.

Taking the functional derivative with respect to $\mathbf{f}(\mathbf{x}')$ of Eqs. (3.18) and (3.19) (for $\mathbf{x}' \neq \mathbf{x}$) yields

$$\nabla \times \nu \nabla^2 \frac{\delta^2 \Phi}{\delta \mathbf{f}(\mathbf{x})\, \delta \mathbf{f}(\mathbf{x}')} = -\nabla \times \frac{\delta^2 \Phi}{\delta \mathbf{h}(\mathbf{x})\, \delta \mathbf{f}(\mathbf{x}')} \tag{3.20}$$

$$\nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \frac{\delta^2 \Phi}{\delta \mathbf{g}(\mathbf{x})\, \delta \mathbf{f}(\mathbf{x}')} \right) = 0 \tag{3.21}$$

Substituting the ansatz

$$\frac{\delta \Phi}{\delta f_j(\mathbf{x}')} = G_j \left[\int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{B}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) + \mathbf{p}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) + \mathbf{q}(\mathbf{x}) \cdot \mathbf{h}(\mathbf{x}), \mathbf{x}' \right] \tag{3.22}$$

(where G_j is an arbitrary functional), we obtain

$$\begin{aligned} \mathbf{q}(\mathbf{x}) &= -\nu \nabla^2 \mathbf{B}(\mathbf{x}) + \nabla C(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \times \mathbf{p}(\mathbf{x}) &= \nabla A(\mathbf{x}) \end{aligned} \tag{3.23}$$

where $C(\mathbf{x})$ and $A(\mathbf{x})$ are arbitrary. This yields

$$\frac{\delta\Phi}{\delta f_j(\mathbf{x}')} = G_j\left[\int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{B}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}) + \left[\alpha\mathbf{B}(\mathbf{x}) - \frac{\mathbf{B}(\mathbf{x})}{|\mathbf{B}(\mathbf{x})|^2}\times\nabla A(\mathbf{x})\right]\cdot\mathbf{g}(\mathbf{x}) + \left[-\nu\nabla^2\mathbf{B}(\mathbf{x}) + \nabla C(\mathbf{x})\right]\cdot\mathbf{h}(\mathbf{x}),\, \mathbf{x}'\right] \tag{3.24}$$

where α is a scalar field to be determined and $\mathbf{B}(\mathbf{x})$ is chosen to be orthogonal to $\nabla A(\mathbf{x})$.

One may verify that

$$\alpha(\mathbf{x}) = \frac{\langle u_j \boldsymbol{\omega}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\rangle}{\langle u_j \mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\rangle} \tag{3.25}$$

For flow localized in a narrow range of wavenumbers about k, α may be viewed as the ratio of helicity and energy currents in k-space, since (as a crude estimate[13]) $[\partial_t k]/k \sim -[\partial_t \tilde{u}(k)]/\tilde{u}(k)$ by incompressibility $\sim k\tilde{u}(k)$, so that $\partial_t(1/k) \sim \tilde{u}(k)$. Negative α, for example, would be consistent with opposite energy and helicity cascades.[14] One expects α to be proportional to the inverse of the integral length scale.

Similarly, one may verify that

$$\frac{\nabla_n C}{\nabla^2 B_n} = \frac{\langle u_j(F_n - \nu\nabla^2 u_n)\rangle}{\langle u_j \nabla^2 u_n\rangle} \tag{3.26}$$

In order to ensure homogeneity of velocity statistics, we choose a solution of the form (3.22) with

$$\begin{gathered} \mathbf{B}(\mathbf{x}) \sim \mathbf{b}_m \exp(i\mathbf{k}_m\cdot\mathbf{x}) \\ G_j[z, \mathbf{x}'] = \sum_m \left(\frac{a_{mj}}{b_{mj}}\right) \ln[\exp(i\mathbf{k}_m\cdot\mathbf{x}') - z] \end{gathered} \tag{3.27}$$

With this choice, homogeneity of vorticity and force statistics requires that

$$\begin{gathered} \alpha = \text{const} \equiv \alpha_m \\ A(\mathbf{x}) \sim A_m \exp(i2\mathbf{k}_m\cdot\mathbf{x}) \\ C(\mathbf{x}) \sim C_m \exp(i\mathbf{k}_m\cdot\mathbf{x}) \end{gathered} \tag{3.28}$$

Hence, substituting into (3.24), the argument in the above expression is given by

$$z = \int d\mathbf{x}\ [\exp(i\mathbf{k}_m \cdot \mathbf{x})] \left\{ \mathbf{b}_m \cdot \mathbf{f}(\mathbf{x}) + \left[\alpha_m \mathbf{b}_m - 2i\mathbf{b}_m \times \frac{\mathbf{k}_m A_m}{|\mathbf{b}_m|^2} \right] \cdot \mathbf{g}(\mathbf{x}) \right.$$
$$\left. + [\nu k_m^2 \mathbf{b}_m + i\mathbf{k}_m C_m] \cdot \mathbf{h}(\mathbf{x}) \right\} \tag{3.29}$$

where $\mathbf{b}_m$ is chosen to be orthogonal to $\mathbf{k}_m$.

Given $\delta\Phi/\delta f_j(\mathbf{x}')$ satisfying (3.20) and (3.21), the functional Φ obtained (in principle) by functional integration satisfies (3.18) and (3.19), as may be seen by commuting a functional integration over $f_j(\mathbf{x}')$ back in through the other operators acting on $\delta\Phi/\delta f_j(\mathbf{x}')$ in (3.20) and (3.21) and setting the arbitrary constants of the functional integration [functions independent of $f_j(\mathbf{x}')$] equal to zero. Hence (3.22), (3.27), (3.29) constitute an implicit solution of the Hopf equation *and* give explicit statistics.

This leads to mean velocity

$$\langle u_j(\mathbf{x}) \rangle = \sum_m \left(\frac{a_{mj}}{b_{mj}} \right) (\mathbf{k}_m \cdot \mathbf{x}) \tag{3.30}$$

One might consider adding a term of the form (2.14) (with **g** replaced by **f**) to Φ. However, although the Hopf equation *is* invariant under this operation, the additional contributions to the mean velocity and strain rate are unphysical since they contain no energy [as may be verified by functional-differentiating (2.14) twice with respect to f]. Hence we choose to disregard this spurious "inhomogeneous Galilean" invariance. Applying incompressibility and the requirement of zero mean shear imposes seven further constraints upon the $5N$ remaining coefficients in $\mathbf{a}_m$ and $\mathbf{b}_m$, where N is the number of wavevectors in our expansion (3.27).

The velocity–force correlation function takes the form

$$\langle u_j(\mathbf{x})\, F_j(\mathbf{x}') \rangle = \sum_m a_{mj} \left[\nu k_m^2 + i \frac{k_{mj}}{b_{mj}} C_m \right] \exp[i\mathbf{k}_m \cdot (\mathbf{x} - \mathbf{x}')] \tag{3.31}$$

One may compare this with the mean transfer into vector component E_j of the energy

$$\varepsilon_{jnp} \langle u_j(\mathbf{x})\, u_n(\mathbf{x}')\, \omega_p(\mathbf{x}') \rangle = \sum_m \frac{a_{mj}}{b_{mj}} [2k_{mj} A_m] \exp[i2\mathbf{k}_m \cdot (\mathbf{x} - \mathbf{x}')] \tag{3.32}$$

(no sum over j). The associated autocorrelation function

$$\langle u_j(\mathbf{x})\, u_j(\mathbf{x}')\rangle = \sum_m a_{mj} \exp[i\mathbf{k}_m \cdot (\mathbf{x}-\mathbf{x}')] \tag{3.33}$$

tells us that $a_{mj} = \langle |\tilde{u}_j(\mathbf{k}_m)|^2\rangle$. From this, we see that the role of the factor of 2 in the exponential on the right-hand side of the nonlinear-transfer term (3.32) is to generate the cascade; energy initially localized in k-space around $\mathbf{k}_m$ will give rise to a transfer of energy to $2\mathbf{k}_m$, which in turn results in transfer to $4\mathbf{k}_m$ and so on. Of course, because the sum of the nonlinear terms vanishes for this class of flows, there is no *net* cascade. In fact, from (3.39) and (3.32), we see that transfer due to Coriolis forces cancels the transfer due to the gradient of the kinetic energy, implying that velocity and pressure gradient are uncorrelated for these flows. This suggests that the *statistical* fixed point of the forced Navier–Stokes equation which corresponds to the deterministic fixed point of the Euler equation may in fact be stable since there is no pressure-driven tendency to isotropize the angle between $\tilde{\mathbf{u}}(\mathbf{k}) \times \tilde{\boldsymbol{\omega}}(\mathbf{k})$ and $\mathbf{k}$. The fixed point is statistical because (i) the statistics are stationary, whereas the flow field in any individual realization may not be, (ii) the correlation functions obtained do not factor as a deterministic correlation function would, and (iii) the "Eulerization" constraint is only imposed weakly.

Three-point correlations may also be derived, e.g.,

$$\langle u_j(\mathbf{x})\, u_n(\mathbf{x}')\, u_p(\mathbf{x}'')\rangle = \sum_m \frac{a_{mj}}{b_{mj}} b_{mn} b_{mp} \exp[i\mathbf{k}_m \cdot (\mathbf{x}+\mathbf{x}''-2\mathbf{x}')] \tag{3.34}$$

Symmetry then implies that

$$a_{mj} = b_{mj}^2 \tag{3.35}$$

Writing $(\mathbf{x}+\mathbf{x}''-2\mathbf{x}')$ as $(\mathbf{x}-\mathbf{x}')+(\mathbf{x}''-\mathbf{x}')+(\mathbf{x}'-\mathbf{x}')$ and using homogeneity to translate the origin by $\mathbf{x}'$ yields a manifestly symmetric form for (3.34). Equivalently, a necessary condition for (3.34) to be symmetric is that $\mathbf{x}'=0$; however, for a homogeneous system, this condition can always be satisfied by translation. (For a more rigorous treatment including sufficiency, see Appendix.) C_m is constrained to vanish, as may be seen by computing the correlation of any product of velocities with both sides of the stationary Navier–Stokes equation and substituting (3.31)–(3.33). However, force–force statistics are still undetermined; the external force may have an arbitrary component which is uncorrelated with $\mathbf{u}$ as well as a component satisfying (3.31), e.g., white noise.[15]

3.3. Homogeneous Steady Solution with Forcing

Let us extend this solution to the case of *general* balance in which (3.18) and (3.19) are not individually valid, but their *sum* is. Then (3.23) becomes

$$\mathbf{q}(\mathbf{x}, \mathbf{x}') = -\nu\nabla^2\mathbf{B}(\mathbf{x}) + \nabla C(\mathbf{x}) - \mathbf{H}(\mathbf{x})\frac{G_j''(z, \mathbf{x}')}{G_j'(z, \mathbf{x}')}$$
$$\mathbf{B}(\mathbf{x}) \times \mathbf{p}(\mathbf{x}) = \nabla A(\mathbf{x}) + \mathbf{H}(\mathbf{x}) \tag{3.36}$$

where $\mathbf{H}(\mathbf{x})$ is not a gradient and the primes on G_j denote derivative with respect to z. Without loss of generality, the potential component of $\mathbf{H}(\mathbf{x})$ may be absorbed into the definitions of C and A. Then homogeneity implies that

$$\mathbf{H}(\mathbf{x}) \sim \mathbf{H}_m \exp(i2\mathbf{k}_m \cdot \mathbf{x}) \tag{3.37}$$

where $\mathbf{H}_m$ is orthogonal to $\mathbf{k}_m$. Since by incompressibility $\mathbf{b}_m$ is orthogonal to $\mathbf{k}_m$ and hence to $\mathbf{H}_m$ [by (3.36)], we obtain that the three vectors $\mathbf{b}_m$, $\mathbf{k}_m$, and $\mathbf{H}_m$ form an orthogonal triad. The self-consistency requirement $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ then implies that

$$i\mathbf{k}_m \times \mathbf{b}_m = \alpha_m \mathbf{b}_m - \mathbf{b}_m \times \frac{(2i\mathbf{k}_m A_m + \mathbf{H}_m)}{|\mathbf{b}_m|^2} \tag{3.38}$$

This can be satisfied if

$$|\mathbf{b}_m|^2 = 2A_m \tag{3.39}$$

$$\alpha_m \mathbf{b}_m = -\mathbf{H}_m \times \frac{\mathbf{b}_m}{|\mathbf{b}_m|^2} \tag{3.40}$$

$$\mathbf{H}_m \cdot \mathbf{b}_m = 0 \tag{3.41}$$

(3.39) fixed the normalization of $\mathbf{b}_m$. Equation (3.40) and (3.41) are satisfied for nonzero α_m if $\alpha_m \mathbf{b}_m$ is chosen to be the vector Fourier coefficient at "wavevector" $i\mathbf{H}_m/\alpha_m$ ($\mathbf{H}_m$ chosen such that $\mathbf{H}_m/\alpha_m$ is imaginary) of any hypothetical incompressible Arnold–Beltrami–Childress flow $\mathbf{v}(\mathbf{r})$ with $\nabla \times \mathbf{v}(\mathbf{r}) = 2A_m \mathbf{v}(\mathbf{r})$.

$\mathbf{H}$ plays the roll of a rotational stirring force; the number of nonzero coefficients $\mathbf{H}_m/\alpha_m$ is a measure of the nonlinearity of the flow, i.e., deviation of the flow from the case of fully-depressed nonlinearity in which $\mathbf{u}(\mathbf{x}) \times \boldsymbol{\omega}(\mathbf{x}) = \nabla A(\mathbf{x})$. If all of the $\mathbf{H}_m$ vanish, we recover the balance discussed in the previous section, in which A_m is given by (3.39) and $\alpha_m = 0$.

The nonlinear transfer (3.32) now becomes

$$\varepsilon_{jnp}\langle u_j(\mathbf{x})\, u_n(\mathbf{x}')\, \omega_p(\mathbf{x}')\rangle = \sum_m \frac{a_{mj}}{b_{mj}} [2k_{mj} A_m - iH_{mj}] \exp[i2\mathbf{k}_m \cdot (\mathbf{x} - \mathbf{x}')] \tag{3.42}$$

$\mathbf{k}_m$ is constrained to be orthogonal to $\mathbf{H}_m$ and $\mathbf{b}_m$. Equations (3.22), (3.27), and (3.29) with the right-hand side of (3.38) replacing the corresponding expression in (3.29) then constitutes a homogeneous, stationary, incompressible, and self-consistent closed solution of the Hopf equation [if there exists a force $\mathbf{F}(\mathbf{x})$ which gives rise to homogeneous stationary flow].

In the absence of homogeneity (but with forcing), Eq. (3.38) is replaced by the condition (3.24) with the modification that $\mathbf{H}$ is added onto ∇A, where

$$\mathbf{B} \text{ satisfies the boundary conditions on } \mathbf{u} \tag{3.43}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.44}$$

A and $\mathbf{H}$ are fixed by self-consistency. For example, setting $\alpha = 0$ yields

$$\nabla A + \mathbf{H} = \mathbf{B} \times (\nabla \times \mathbf{B}) \tag{3.45}$$

satisfied by

$$\mathbf{H} = \mathbf{B} \cdot \nabla \mathbf{B} \tag{3.46}$$

$$A = |\mathbf{B}|^2/2 \tag{3.47}$$

Dropping the $\mathbf{x}'$ dependence on the right-hand side of the modified (3.24) also yields a possible expression for, not $\delta\Phi/\delta f_j(\mathbf{x}')$, but Φ itself (except for the case of homogeneous statistics). Again, the resulting statistics are (by construction) consistent with the boundary conditions as well as with stationary, incompressibility, and self-consistency.

3.4. Inhomogeneous Steady Solution with Boundary Conditions

For the unforced inhomogeneous case, the above may be simplified by returning to the velocity–vorticity characteristic functional (2.4), where

$$\frac{\delta\Phi}{\delta g_j} = G_j\left(\frac{i}{\nu} H(\mathbf{x}) - \int d\mathbf{x}'\, \mathbf{f}(\mathbf{x}') \cdot [\nabla H(\mathbf{x}') + \mathbf{M}(\mathbf{x}')]\right) \tag{3.48}$$

$\nabla H + \mathbf{M}$ is chosen to be solenoidal and satisfy boundary conditions on $\mathbf{u}$. Noting that

$$\nu\nabla^2\boldsymbol{\omega} = -\nu\nabla\times\nabla\times\boldsymbol{\omega} \tag{3.49}$$

the condition for stationarity becomes

$$\nabla\times\left(i\frac{\delta}{\delta\mathbf{f}}+\nu\nabla\right)\times\frac{\delta\Phi}{\delta\mathbf{g}}=0 \tag{3.50}$$

[compare with (3.19)], implying

$$\nabla\times[\mathbf{M}(\mathbf{x})\times\mathbf{G}'(z(\mathbf{x}))]=0 \tag{3.51}$$

where the prime denotes derivative of $\mathbf{G}$ with respect to its argument $z(\mathbf{x})$ [given in parentheses in (3.48)], not to be confused with the "del" (∇), which as usual denotes derivative with respect to $\mathbf{x}$. Stationarity may then be achieved by choosing $\mathbf{G}'$ to be parallel to $\mathbf{M}$, or more generally, for

$$\mathbf{G}'\cdot\nabla\mathbf{M}-\mathbf{M}\cdot\nabla\mathbf{G}'+\mathbf{M}\nabla\cdot\mathbf{G}'-\mathbf{G}'\nabla\cdot\mathbf{M}=0 \tag{3.52}$$

The longitudinal counterpart of (3.50) determines the steady-state pressure. An analogous result holds for the induction equation[11] in magnetohydrodynamics.

The requirement that the mean vorticity be the curl of a mean velocity can be satisfied only if

$$\nabla\cdot\mathbf{G}=0 \qquad \text{at} \quad \mathbf{f}=0 \tag{3.53}$$

Noting

$$\nabla\cdot\mathbf{G}=\mathbf{G}'\cdot\nabla H \tag{5.54}$$

yields

$$\mathbf{G}'=\mathbf{b}\times\nabla H \tag{3.55}$$

for some vector field $\mathbf{b}$. The spatial dependence of $\mathbf{b}$ may be determined by noting that solenoidality of vorticity requires

$$\nabla\cdot\frac{\partial^n\mathbf{G}(\mathbf{f}=0)}{\partial H^n}=0 \tag{3.56}$$

Linearity of the Hopf equation allows us to generalize (3.48) to

$$\frac{\delta\Phi}{\delta g_j}=\sum_q a_q G_{qj}\left(\frac{i}{\nu}H_q(\mathbf{x})-\int d\mathbf{x}'\,\mathbf{f}(\mathbf{x}')\cdot[\nabla H_q(\mathbf{x}')+\mathbf{M}_q(\mathbf{x}')]\right) \tag{3.57}$$

In general, we may write

$$G_{qj}(z_q(\mathbf{x});\mathbf{f}=0)=\sum_p A_{qj}(p)e^{(p/\nu)H_q(\mathbf{x})} \tag{3.58}$$

where the H_q are bounded. Then (3.56) implies

$$0 = p^n \mathbf{A}_q(\mathbf{p}) \cdot \nabla H_q(\mathbf{x}) \tag{3.59}$$

which will be satisfied if $\nabla H_q(\mathbf{x})$ lies in a plane for all $\mathbf{x}$ and the $\mathbf{A}_q(p)$ are normal to that plane for all p.

The additivity of probabilities implied by the linearity of the Hopf equation suggests that (3.57) may be interpreted as a decomposition of the flow into statistically orthogonal (mutually exclusive) states. The vorticity associated with each state is arbitrarily aligned but uniaxial (i.e., different from state to state, but everywhere parallel or antiparallel within any given state). Of course, the sum over states yields a mean vorticity whose direction may vary in space, as is generally desired. The sum over states also implies that the correlation functions in general do not factor (unless there is only one H_q and each G_{qj} happens to be exponential in H_q). In other words, we have a true statistical solution rather than a deterministic solution; the vorticity associated with each state (and the mean vorticity) need not satisfy the curl of the Navier–Stokes equation. (For example, a set of H_q and $\mathbf{M}_q$ can be found that would correspond to a representation of the flow as an ensemble of vortex filaments of varying core diameters; the a_q would then be given by the Bose distribution.[16]) This allows us to identify those coherent structures which characterize the *ensemble*, rather than particular realizations.[17]

Explicitly, (3.57) becomes

$$\frac{\delta \Phi}{\delta g_j(\mathbf{x})} = \sum_{q,p} a_q A_{qj}(p) \exp\left(\frac{p}{\nu} H_q(\mathbf{x}) + \int d\mathbf{x}'\, ip\mathbf{f}(\mathbf{x}') \cdot [\nabla H_q(\mathbf{x}') + \mathbf{M}_q(\mathbf{x}')]\right) \tag{3.60}$$

implying

$$\langle \omega_j(\mathbf{x}) \rangle = \sum_{q,p} a_q A_{qj}(p) \exp\left(\frac{p}{\nu} H_q(\mathbf{x})\right) \tag{3.61}$$

$$\langle \omega_j(\mathbf{x})\, u_n(\mathbf{x}') \rangle = \sum_{q,p} a_q \frac{p}{\nu} A_{qj}(p) \exp\left(\frac{p}{\nu} H_q(\mathbf{x})\right) [\nabla_n H_q(\mathbf{x}') + M_{qn}(\mathbf{x}')] \tag{3.62}$$

$$\langle \omega_j(\mathbf{x})\, \omega_{j'}(\mathbf{x}') \rangle = \sum_{q,p} a_q \frac{p}{\nu} A_{qj}(p) \exp\left[\frac{p}{\nu} H_q(\mathbf{x})\right] [\nabla \times M_q]_{j'}(\mathbf{x}') \tag{3.63}$$

etc.

Velocity autocorrelation functions may be obtained by applying Biot-Savart to the velocity–vorticity correlation functions. Alternatively, we see that from $\langle \mathbf{u}(\mathbf{x})\,\mathbf{u}(\mathbf{x}')\rangle$, one may take the curl to find (3.62) and hence $\langle \mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}')\times\boldsymbol{\omega}(\mathbf{x}')\rangle$, thereby effecting a closure for unforced 3D Navier–Stokes flow with arbitrary boundary conditions.

Positivity of the energy spectrum implies that the Fourier transform of (3.63) with respect to $\mathbf{x}$ and $\mathbf{x}'$ must be nonnegative. This imposes a restriction on the coefficients a_q appearing in (3.60).

Symmetry under interchange of $\mathbf{x}$, j and $\mathbf{x}'$, j' implies

$$\mathbf{G}'_q(\mathbf{x}) = \sum_p \frac{p}{v}\,\mathbf{A}_q(p)\exp\left(\frac{p}{v}H_q(\mathbf{x})\right) = \nabla\times\mathbf{M}_q(\mathbf{x}) \tag{3.64}$$

If we denote

$$\mathbf{B}_q(\mathbf{x}) = \nabla H_q(\mathbf{x}) + \mathbf{M}_q(\mathbf{x}) \tag{3.65}$$

we find that

$$\nabla\times\mathbf{B}_q(\mathbf{x}) = \mathbf{G}'_q(\mathbf{x}) \tag{3.66}$$

where the basis functions $\mathbf{B}_q(\mathbf{x})$ are solenoidal (which fixes ∇H_q, given $\mathbf{M}_q$) and satisfy the same boundary conditions as $\mathbf{u}$. Note that $\hat{\zeta}_q\cdot\mathbf{M}_q(\mathbf{x}) = \hat{\zeta}_q\cdot\mathbf{B}_q(\mathbf{x})$, where the unit vector $\hat{\zeta}_q$ is defined to be along $\mathbf{A}_q$, with mutually perpendicular vectors $\hat{\xi}_q$ and $\hat{\eta}_q$.

Hence, given $\mathbf{B}_q$, we can obtain $\mathbf{G}'_q$ by invoking symmetry; the problem now becomes to find the coefficients a_q (by using stationarity) so that we may write down expressions for the two-point moments, given one-point moments. The stationarity conditions may be written as

$$\nabla_\perp\cdot\mathbf{M}_q(\mathbf{x}) = -\nabla_\perp \ln|\mathbf{G}'_q|\cdot\mathbf{M}_q(\mathbf{x}) \tag{3.67}$$

$$\hat{\zeta}_q\cdot\nabla[\hat{\eta}_q\cdot\mathbf{M}_q(\mathbf{x})] = \hat{\zeta}_q\cdot\nabla[\hat{\xi}_q\cdot\mathbf{M}_q(\mathbf{x})] = 0 \tag{3.68}$$

where $\nabla_\perp \equiv (\hat{\xi}_q\cdot\nabla,\ \hat{\eta}_q\cdot\nabla,\ 0)$. We can use (3.59) and the fact [from (3.64)] that $\nabla\times\mathbf{M}_q(\mathbf{x})$ only has a ζ_q component to deduce that $\hat{\zeta}_q\cdot\mathbf{B}_q$ only depends upon ζ_q while $\hat{\xi}_q\cdot\mathbf{B}_q$ and $\hat{\eta}_q\cdot\mathbf{B}_q$ do not depend upon ζ_q. In the 2D case, we obtain a generalization of (2.5).

Changing variables to

$$\mathbf{M}_q(\mathbf{x}) = \mathbf{M}^0_q(\mathbf{x})/|\mathbf{G}'_q(\mathbf{x})| \tag{3.69}$$

we find that Eq. (3.67) becomes

$$\nabla_\perp\cdot\mathbf{M}^0_q(\mathbf{x}) = 0 \tag{3.70}$$

This implies that we may write

$$\hat{\xi}_q \cdot \mathbf{M}_q^0(\mathbf{x}) \equiv \hat{\eta}_q \cdot \nabla \Phi_q, \qquad \hat{\eta}_q \cdot \mathbf{M}_q^0(\mathbf{x}) \equiv -\hat{\xi}_q \cdot \nabla \Phi_q \tag{3.71}$$

for some $\Phi_q(\xi_q, \eta_q)$. Substituting (3.70) and (3.71) into (3.64) then yields

$$\nabla^2 \Phi_q - \nabla \ln |\mathbf{G}_q'| \cdot \nabla \Phi_q + |\mathbf{G}_q'|^2 = 0 \tag{3.72}$$

Solving for Φ_q and using (3.69) and (3.71) then yields $\mathbf{M}_q(\mathbf{x})$.

From (3.65) we obtain ∇H_q; multiplying by $\mathbf{G}_q'$ and integrating yields $\mathbf{G}_q$. Given one-point moments, we may expand them in terms of $\mathbf{G}_q$ to obtain the coefficients a_q, which finally may be substituted into (3.62) to give us the two-point moments.

One drawback with this approach is that although the basis functions $\mathbf{B}_q$ with which we start may be orthogonal, the resulting $\mathbf{G}_q$ in which we expand the one-point moments may not in general be orthogonal. Hence we approach the problem from the other end: given the one-point moment $\langle \omega_j(\mathbf{x}) \rangle$ (and its expansion coefficients a_q in terms of orthogonal functions $\mathbf{G}_q$), find the basis functions $\mathbf{B}_q$ and $\mathbf{G}_q'$ so that we may write down two-point moments such as $\langle \omega_j(\mathbf{x})\, u_n(\mathbf{x}') \rangle$. To do this, note that (3.69) and (3.71) may be written as

$$(B_{qi} - \nabla_i H_q)\, \nabla_i(\hat{\zeta}_q \cdot \mathbf{G}_q) = (\nabla \Phi_q \times \hat{\zeta}_q)_i\, \nabla_i H_q \tag{3.73}$$

(no sum over i). This may be solved to obtain

$$\nabla_i H_q = B_{qi} \left[1 + \frac{(\nabla \Phi_q \times \hat{\zeta}_q)_i}{\nabla_i(\hat{\zeta}_q \cdot \mathbf{G}_q)} \right]^{-1} \tag{3.74}$$

while (3.66) becomes

$$\hat{\zeta}_q \cdot \{\nabla \times \mathbf{B}_q\} = \frac{\nabla_i(\hat{\zeta}_q \cdot \mathbf{G}_q)}{\nabla_i H_q} \tag{3.75}$$

($i = \xi_q, \eta_q$, no sum implied). Together with the solenoidality condition

$$\nabla \cdot \mathbf{B}_q(\mathbf{x}) = 0 \tag{3.76}$$

we have three equations for the three unknown fields $(\hat{\xi}_q \cdot \mathbf{B}_q)$, $(\hat{\eta}_q \cdot \mathbf{B}_q)$, and Φ_q. Note that $(\hat{\zeta}_q \cdot \mathbf{B}_q)$ is an arbitrary function of ζ_q (it does not appear in either the stationarity or symmetry conditions) and is constrained only by the boundary conditions on $(\hat{\zeta}_q \cdot \mathbf{u}(\mathbf{x}))$. Separation of variables then yields

$$\nabla_\perp \cdot \mathbf{B}_q(\mathbf{x}) = \text{const} \tag{3.77}$$

The uniaxial decomposition reduces to an ordinary Fourier transform if H_q is linear in $\mathbf{x}$ (i.e., $\mathbf{q} \cdot \mathbf{x}$); more generally, the $(p/v) H_q$ may be chosen to be the logarithms of a set of complete orthogonal functions suitable for decomposition of the mean vorticity.

4. CONCLUSIONS

We have reduced the stationary turbulence closure problem, given general boundary conditions (and presumably inhomogeneous statistics), to the problem of solving three coupled first-order nonlinear differential equations. This offers us an exact method for computing two- and higher-point moments, given one-point moments. Many examples remain to be worked out and tested against results from simulation studies. Work is underway on plane channel flow, Boussinesq and compressible flows and will be presented in forthcoming papers.

We have also found closed solutions to the problem of homogeneous forced stationary turbulence. As an example, we have derived a solution exhibiting depletion of nonlinearity, not inconsistent with recent findings. These solutions, however, are less readily compared with experiments, due to the difficulty of computing force–force statistics from force–velocity statistics. One would have to solve the coupled equations[18] for the velocity–force response function and the velocity–velocity correlation function, which may be nontrivial even if given the latter.

We have also derived other, more specialized solutions to the stationary Hopf equation (e.g., in the presence of mean uniform shear, as well as operator or matrix solutions) whose physical significance, if any, remains to be clarified. Further intriguing longer-range questions include: (i) nonuniqueness[10] of solutions, their selection mechanism and stability, (ii) the feasibility of inverse-functional Fourier transforming Φ to obtain the steady-state velocity probability density function (pdf) (which one certainly hopes will turn out to be positive), (iii) the possibility of incorporating initial conditions and time dependence (to find two-time correlations), and (iv) the actual prediction (rather than assumption) of one-point statistics from the boundary conditions (by substituting the mean Reynolds stress, computed from the mean velocity, back into the mean of the stationary Navier–Stokes equation.)

APPENDIX

Explicit symmetrization of the correlation functions for the homogeneous forced case may be obtained by generalizing (3.22) to

$$\frac{\delta\Phi}{\delta f_m(\mathbf{x})} \equiv G_m[z, \mathbf{x}] + \int_{-\infty}^{\infty} d\mathbf{x}'\, f_j(\mathbf{x}')\, g_{jm}^{(1)}(\mathbf{x}, \mathbf{x}') + \int_{-\infty}^{\infty} d\mathbf{x}'\, d\mathbf{x}''\, f_j(\mathbf{x}')\, f_n(\mathbf{x}'')\, g_{jnm}^{(2)}(\mathbf{x}, \mathbf{x}', \mathbf{x}'') + \cdots \tag{A1}$$

We then impose

$$g_{jm}^{(1)}(\mathbf{x}, \mathbf{x}') = \frac{1}{2}\left[\frac{\delta G_j(\mathbf{x}')}{\delta f_m(\mathbf{x})} - \frac{\delta G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')}\right]_{\mathbf{f}=\mathbf{g}=\mathbf{h}=0} \tag{A2}$$

$$g_{jnm}^{(2)}(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = \frac{1}{3}\left[\frac{\delta^2 G_j(\mathbf{x}')}{\delta f_m(\mathbf{x})\,\delta f_n(\mathbf{x}'')} - 2\,\frac{\delta^2 G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')\,\delta f_n(\mathbf{x}'')} + \frac{\delta^2 G_n(\mathbf{x}'')}{\delta f_m(\mathbf{x})\,\delta f_j(\mathbf{x}')}\right]_{\mathbf{f}=\mathbf{g}=\mathbf{h}=0} \tag{A2$'$}$$

etc. Correlation functions then become manifestly symmetric, e.g.,

$$\begin{aligned} &\langle u_j(\mathbf{x})\, u_n(\mathbf{x}')\, u_p(\mathbf{x}'')\rangle \\ &= \sum_m b_{mj} b_{mn} b_{mp} \\ &\quad \times \{\exp[i\mathbf{k}_m \cdot (\mathbf{x} + \mathbf{x}'' - 2\mathbf{x}')] + \exp[i\mathbf{k}_m \cdot (\mathbf{x} + \mathbf{x}' - 2\mathbf{x}'')] \\ &\quad + \exp[i\mathbf{k}_m \cdot (\mathbf{x}' + \mathbf{x}'' - 2\mathbf{x})]\} \end{aligned} \tag{A3}$$

More generally, Φ is invariant under transformations upon G of the form

$$\left[\frac{\delta G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')}\right] \to \left[\frac{\delta G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')} + \frac{\delta F_m^{(1)}(\mathbf{x})}{\delta f_j(\mathbf{x}')} - \frac{1}{2}\left\{\frac{\delta F_m^{(1)}(\mathbf{x})}{\delta f_j(\mathbf{x}')}\right\}\right] \tag{A4}$$

$$\left[\frac{\delta^2 G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')\,\delta f_n(\mathbf{x}'')}\right] \to \left[\frac{\delta^2 G_m(\mathbf{x})}{\delta f_j(\mathbf{x}')\,\delta f_n(\mathbf{x}'')} + \frac{\delta^2 F_m^{(2)}(\mathbf{x})}{\delta f_j(\mathbf{x}')\,\delta f_n(\mathbf{x}'')} - \frac{1}{3}\left\{\frac{\delta^2 F_m^{(2)}(\mathbf{x})}{\delta f_j(\mathbf{x}')\,\delta f_n(\mathbf{x}'')}\right\}\right] \tag{A4$'$}$$

etc., where the braces denote summation over all permutations of the position arguments (carrying the vector subscripts along with the corresponding arguments) and the $F_m^{(i)}(\mathbf{x})$ are arbitrary functionals of $\mathbf{f}(\mathbf{x})$. Using (3.27), (3.29), and the orthogonality of $\mathbf{b}_m$ and $\mathbf{k}_m$, we find that a sufficient condition for stationarity (3.20), (3.21) is

$$\nabla^2 g_{jm\cdots}^{(k)}(\mathbf{x}, \mathbf{x}', \ldots) = 0 \tag{A5}$$

which constrains the $\mathbf{b}_m$ appearing in the correlation functions. Alternatively, impose (A1) and (A2) with $\mathbf{f}$'s replaced by $\mathbf{g}$'s on both sides of the equations and $\mathbf{G}$ replaced by $\nabla \times \mathbf{G}$. z is replaced by the original z plus a contribution $\int \mathbf{g} \cdot \nabla \times \mathbf{M}_q$. This manifestly satisfies stationarity and symmetry; self-consistency is imposed by defining velocity correlation functions via Biot-Savart. This also obviates the need for a subsidiary condition such as (A5).

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